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# THE METHOD OF CONTOUR DYNAMICS FOR AXISYMMETRIC VORTICAL STRUCTURES WITH SPATIALLY BOUNDED VORTICITY*

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The method of contour dynamics, used in /1/ to describe the interaction of plane vortices, is extended here to axisymmetric vortical flows. Numerical modelling of the evolution of the simplest contours is used as the basis for proposing a hypothesis of the existence of periodic, vortical soliton-like structures.

We base our analysis on the equations of an ideal incompressible fluid written in the form

$$\partial\Omega/\partial t + \operatorname{rot}[\Omega \mathbf{v}] = 0, \quad \Omega = \operatorname{rot}\mathbf{v}, \quad \operatorname{div}\mathbf{v} = 0 \tag{1}$$

Here $\mathbf{v}$ is the velocity field and $\Omega$ is the vorticity field. In what follows we shall consider only axisymmetric flow for which $\Omega = \{0, \rho\omega(\rho, z), 0\}$ inside certain regions and $\Omega = 0$ outside them, using a cylindrical system of coordinates. Taking this into account, we shall rewrite the first equation of (1) in the form

$$\frac{\partial\omega}{\partial t} + v_z\frac{\partial\omega}{\partial z} + v_\rho\frac{\partial\omega}{\partial\rho} = 0 \tag{2}$$

We shall assume that $\omega(\rho, z)$ is a piecewise constant function, i.e.

$$\omega(\rho, z) = \sum_{i=1}^{N} \gamma_i\Theta(S_i) \tag{3}$$

Here $\gamma_i = \text{const}$ characterizes the intensity of the vortices, and $\Theta(S_i)$ is the characteristic function of the singly connected region $S_i$ whose boundary is described by the parametric equations

$$\rho = \rho_i(s_i, t), \; z = z_i(s_i, t) \tag{4}$$

where $s_i$ is a parameter varying along the boundary. In this case we can show that Eq.(2) is satisfied identically if

$$\partial\rho_i/\partial t = v_\rho(\rho_i, z_i), \quad \partial z_i/\partial t = v_z(\rho_i, z_i) \tag{5}$$

In fact, Eqs.(5) always hold, since they express the well-known Helmholz theorem.

Thus we have shown that if the distribution of the vorticity is described by the piecewise-constant function (3), it will continue to be described by the same function at subsequent instants of time, and only the boundaries of the regions will vary.

We shall now show that, assuming that representation (3) holds, we can express the velocity field $\mathbf{v}$ in terms of contour integrals over the boundaries of the regions with constant vorticity, using universal integrands.

To do this, we shall consider the last two equations of (1). In accordance with the well-known formulas, the solution of these equations decreasing at infinity, has the form

$$\mathbf{v} = \operatorname{rot}\mathbf{A}, \quad \mathbf{A} = \frac{1}{4\pi}\int d^3\mathbf{r}'\frac{\Omega(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \tag{6}$$

In the axisymmetric case the vector $\mathbf{A}$, just like $\Omega$, has only the azimuthal component $A_\varphi$, so that instead of (6) we obtain

$$A_\varphi = \frac{1}{4\pi}\int_0^\infty \rho'^2 d\rho' \int_{-\infty}^{\infty} dz' \int_0^{2\pi} d\varphi \frac{\omega(\rho', z')\cos\varphi}{[(z-z')^2+\rho^2+\rho'^2-2\rho\rho'\cos\varphi]^{1/2}}$$
$$v_\rho = -\frac{\partial A_\varphi}{\partial z}, \quad v_z = \frac{1}{\rho}\frac{\partial(\rho A_\varphi)}{\partial\rho} \tag{7}$$

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Substituting expression (3) for $\omega$, we obtain

$$A_\varphi = \Sigma \frac{\gamma_i}{4\pi} \int_0^{2\pi} d\varphi \cos\varphi \int_{S_i} d\rho_i\, dz_i \rho_i^2 \frac{\partial}{\partial z_i} \ln (z_i - z + [(z_i - z)^2 + \rho^2 + \rho_i^2 - 2\rho\rho_i \cos\varphi]^{1/2}) \tag{8}$$

Here and henceforth summation is taken from $i = 1$ to $i = N$.

Applying Green's formula to (8), we obtain the expression for $A_\varphi$ in the form of an integral over the boudaries of the region $\partial S_i$. Differentiating this result with respect to $z$, we obtain an expression for $v_\rho$ in the form of a contour integral over the boundaries of the regions with the universal integrand function ($\mathbf{K}(p)$, $\mathbf{E}(p)$ are the complete elliptic integrals of first and second kind respectively)

$$v_\rho = \Sigma \frac{2\gamma_i}{\pi} \oint_{\partial S_i} d\rho_i \rho_i^2 \frac{\mathbf{K}(p) - \mathbf{E}(p)}{r_+ - r_-}$$
$$r_\pm = [(z_i - z)^2 + (\rho_i \pm \rho)^2]^{1/2}, \quad p = (r_+ - r_-)/(r_+ + r_-) \tag{9}$$

Completely analogous but longer reduction also yields ($\Pi$ is an elliptic integral of the third kind)

$$v_z = \Sigma \frac{\gamma_i}{2\pi} \oint_{\partial S_i} d\rho_i \frac{\rho_i}{r_+} \left[\mathbf{K}(k) + \frac{\rho_i - \rho}{\rho_i + \rho} \Pi(n, k)\right](z_i - z)$$
$$k^2 = 1 - r_-^2/r_+^2, \quad n = 4\rho\rho_i/(\rho_i + \rho)^2 \tag{10}$$

Substituting (9) and (10) into system (5), we obtain the final expressions for the equations of contour dynamics for axisymmetric flows of an ideal fluid.

Thus the dynamics of axisymmetric vortical structures is completely determined, under the assumption that vorticity $\omega$ is piecewise constant, by the dynamics of the contours bounding these structures, so that the problem becomes, in fact, one-dimensional.

We note that when flows with a velocity distribution differing from (3) have to be investigated, they can always be approximated by a finite set of piecewise-constant regions. As a result, we can use Eqs.(5), (9) and (10) to describe the dynamics of the vortices with any distribution of vorticity over their cross-section.

Although system (5), (9), (10) is complicated, nevertheless, it can be useful even in analytic investigations. For example, in the case of a Kelvin vortex the asymptotic expansion in $a/R$ ($R$ is the radius of the vortex ring and $a$ is the radius of its cross-section), leads to the well-known expression for the translational velocity of the vortex /2/

$$V = \frac{\varkappa}{4\pi R}\left(\ln\left(\frac{8R}{a}\right) - \frac{1}{4}\right)$$

where $\varkappa$ is the circulation of velocity along the contour. We can find the subsequent terms of the expansion in terms $a/R$ in the same manner.

It should be stressed that system (5), (9), (10) can be used in the search for soliton-like solutions of the equations of hydrodynamics in the axisymmetric case. We can assume in advance that axisymmetric, soliton-like formations of at least two types exist. Firstly, there are vortical formations moving with constant velocity without any change in form. The existence of a series of such solutions was shown in /3/. Secondly, we can assume that solutions exist in which the region of constant vorticity is in translational motion, changing its form at the same time so that after some time $T_0$ we can find a vortical region which reverted to its initial form, but is displaced by some distance along the $z$ axis. The existence of such solutions was shown for the case of the motion of vortex rings in /4, 5/. The results of numerical modelling given below also produce favourable indications that solutions of this type exist.

In the general case, when $\rho(z, t)$ describes the dynamics of the contour, the solution of the second kind can be found by solving the equation

$$\rho(z, t + T_0) = \rho(z - VT_0, t) \tag{11}$$

which should hold for any $t$ and for non-zero $T_0$.

In order to obtain a constructive equation we shall assume that within the class of soliton-like solutions of the second kind, solutions exist which can be represented in the form of a contour rotating about some centre in translational motion. In this case we can obtain, from (11), the following equation:

$$\frac{\partial z(s,t)}{\partial s}(v_\rho - \Omega z(s,t)) = \frac{\partial \rho(s,t)}{\partial s}(v_z - V + \Omega \rho(s,t)) \tag{12}$$

Here $\Omega$ is the constant angular velocity of rotation of the vortex, $V$ is its constant translational velocity, and $v_\rho$ and $v$ are given by (9) and (10).

The solution of Eq.(12), or in more general case, of (11), will give, if it exists for some $V$ and $\Omega$, the form of soliton-like vortices of the second kind.

Let us now return to the simpler problem of constructing a numerical algorithm for solving system (5), (9) and (10) in the case of a single singly connected contour, i.e. for $N=1$. To do this we shall measure all quantities with dimensions of length in units of $R$ ($R$ is the characteristic radius of the vortex), and the quantities with dimensions of time in units of $2\pi/(\gamma R)$. As a result, the dimensionless form of the equations of contour dynamics will take the form

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= 4 \oint_{\partial S} d\rho' \rho'^2 \frac{\mathrm{K}(p) - \mathrm{E}(p)}{r_+ - r_-} \\ \frac{\partial z}{\partial t} &= \oint_{\partial S} d\rho' \rho' \frac{z' - z}{r_+} \left[\mathrm{K}(k) + \frac{\rho' - \rho}{\rho' + \rho} \Pi(n, k)\right]\end{aligned} \tag{13}$$

In what follows, we shall represent the contour in the form of a set of points lying on the contour ($\rho(s_l)$, $z(s_l)$; $s_l = l$; $l = 0, 1, \ldots, L$), and cubic splines approximating the contour between the points of discretization. The approximating splines were given for $l-1 \leqslant s \leqslant l$ in the following form:

$$\begin{aligned}\rho = m_{l-1}(l-s)^2(s-l+1) - m_l(s-l+1)^2(l-s) + \\ \rho(s_{l-1})(l-s)^2[2(s-l)+3] + \rho(s_l)(s-l+1)^2[2(l-s)+1]\end{aligned} \tag{14}$$

(and similarly for $z$). Here $m_l$ and $m_{l-1}$ are the values of the derivatives $\partial\rho/\partial s$ at the nodes $l$ and $l-1$, calculated using four-point formulas. Naturally, (14) must be supplemented by the conditions of periodicity along the contour.

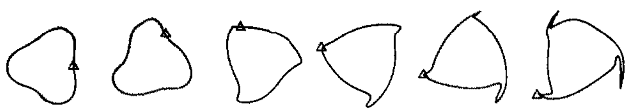

Fig.1

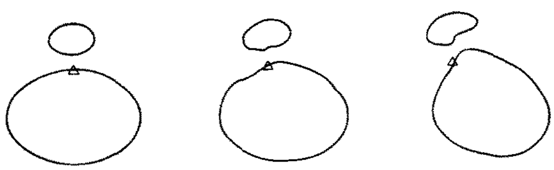

Fig.2

The contour integrals in (13) were calculated using Simpson's formulas, and the middle point was chosen for $s = l - 1/2$ in (14).

Analysing the integrands in (13) we find that they have singularities on the contour of integration. Thus the integrand for $v_\rho$ has a logarithmic divergence at the point $(\rho-\rho')^2 + (z-z')^2 = 0$ and in order to take it into account correctly the integral is evaluated near the singularity analytically. The differential equations with respect to time (13) were solved using the Kutta-Meerson method /6/. The well-known fact that the vortex flow is conserved across the vortex was used as additional control of the correctness of the method.

In order to check Eqs.(13) and their discrete analogue, we first analysed the dynamics of the Hill vortex whose exact solution is known. Computations showed that in this case the right-hand sides of Eqs.(13) are calculated with a relative error of $\varepsilon \sim 10^{-4}$ when the contour is broken into 100 points. Numerical solution of the differential equations with respect to time leads to violation of the theorem of conservation of vorticity $\Phi$ ($\Delta\Phi/\Phi \sim 10^{-3}$ per unit of dimensionless time). The accuracy of this part of the algorithm can easily be increasing the number of points on the contour, but this also increases the calculation time.

After checking the correctness of the numerical scheme on the Hill vortex, we considered the dynamics of a toroidal vortex whose cross-section was bounded by the contour

$$\rho = R + (a + \varepsilon \sin^2 \varphi_0) \sin \varphi, \quad z = (a + \varepsilon \sin^2 \varphi_0) \cos \varphi \tag{15}$$

for $R = 0.8$; $a = 0.1$; $\varepsilon = 0.05$; $0 \leqslant \varphi \leqslant 2\pi$; $\varphi_0 = 3\varphi/2$.

Fig.1 shows the results of the computations for $t = 0$; 0.4; 0.8; 1.2; 1.6; 2.0. In order to follow the rotation of the contour more easily, one of its points has been marked by a triangle. The non-conservation of the vortex flux was monitored during the computations, and it was found to be sufficiently small right up to the maximum time ($t = 2.2$).

We can already observe at $t = 1$ the appearance of kinks on the previously smooth contour. In what follows, the kinks are replaced by vortex filaments which gradually move away from the nucleus of the vortex.

In spite of the appearance and detachment of vortex filaments (they must naturally be represented by a surface), our attention is drawn to the fact that the basic nucleus of the vortex (after the separation of the vortex filaments) rotates as a whole, being at the same time displaced along the $z$ axis. This fact not only allows us to hope that solutions of Eqs. (12) exist, but it obviously also implies their definite stability, manifesting itself in the decomposition of the initial vortex into a soliton-like structure of the second kind described by Eq.(12) and a set of vortex filaments. The final answer to the question of the existence of soliton-like solutions of the equations of contour dynamics can only be given by solving Eq.(12), or the more general Eq.(11).

To illustrate the possibility of using the equations of contour dynamics for several contours, Fig.2 shows the results of computing the evolution of a pair of vortex rings with the same values of $\gamma$, for $t = 0$; 0.1; 0.2.

Because of the reduction in the dimensions of the problem, the method of contour dynamics is fully realizable on a minicomputer. For example, a 32-bit computer performing about $4 \times 10^5$ lengthy operations per second, needs about 30 min to compute a single time step for Fig.1 ($\Delta t = 0.05$)

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